

Intersection of positive closed currents of higher bidegree

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Abstract

Let X be a compact Kähler manifold of dimension n . Let T and S be two positive closed currents on X of bidegree (p, p) and (q, q) respectively with $p + q \leq n$. Assume that T has a continuous super-potential. We prove that the wedge product $T \wedge S$, defined by Dinh and Sibony, is a positive closed current.

Keywords: positive closed current, intersection of currents, super-potential.

1 Introduction

Let X be a compact Kähler manifold of dimension n . Let T and S be two positive closed currents on X of bidegree (p, p) and (q, q) respectively with $p + q \leq n$. In [5], Demailly asked the question to define the intersection $T \wedge S$. The theory of intersections of currents of bidegree $(1, 1)$ is well developed, see, e.g., [1, 3, 4, 10]. So the question of Demailly concerns currents of higher degree.

The problem was recently solved by Dinh and Sibony in [9] using their theory of super-potentials (see also [7]). Assume that T has continuous super-potentials (see [9] or Section 2 for the terminology). Then the wedge product $T \wedge S$ is well-defined. It is known that this product is the difference of two positive closed currents. The operator satisfies basic properties like the commutativity and the associativity when intersect several currents. The Hodge cohomology class of $T \wedge S$ is the cup product of the ones of T and S . Moreover, $T \wedge S$ depends continuously on S . Therefore, it is positive when S can be approximated by smooth positive closed forms. The last property of approximation is satisfied when X is a homogeneous manifold and also in the case of some dynamical Green currents. The purpose of this work is to prove the positivity of $T \wedge S$ in the general setting. We have the following theorem.

Theorem 1.1. *Let X be a compact Kähler manifold of dimension n . Let T and S be two positive closed currents on X of bidegree (p, p) and (q, q) respectively with*

$p+q \leq n$. Assume that T has a continuous super-potential. Then the intersection current $T \wedge S$ is a positive closed current of bidegree $(p+q, p+q)$.

In Section 2, we will recall some basic properties of positive closed currents and their super-potentials. In Section 3, we will introduce an alternative definition of $T \wedge S$ which is a positive closed current. We then show that this definition is equivalent to the one by Dinh and Sibony. The above result will follow immediately. We will present now the main idea.

Suppose first that T and S are positive closed smooth forms of X . Let π_j ($j = 1, 2$) be the projections from $X \times X$ to the first and second components respectively. We have $T \otimes S = \pi_1^*(T) \wedge \pi_2^*(S)$. This is a positive closed smooth form on $X \times X$. Then one can compute $T \wedge S$ via the formula

$$(1.1) \quad T \wedge S = (\pi_j)_*(T \otimes S \wedge [\Delta]) \text{ for } j = 1, 2,$$

where $[\Delta]$ is the current of integration on the diagonal Δ of $X \times X$.

Observe that because of $[\Delta]$, the formula (1.1) can not be extended to general singular currents T and S . We can however use the theory of intersection with $(1, 1)$ -currents if in the place of Δ we have a hypersurface. This is the reason why we consider the blow-up $\widehat{X \times X}$ of $X \times X$ along Δ . Let Π be the natural projection from $\widehat{X \times X}$ to $X \times X$ and $\widehat{\Delta} = \Pi^{-1}(\Delta)$ be the exceptional hypersurface. Recall from [2, 15] that the blow-up of a compact Kähler manifold along a submanifold is also Kähler. Let $\widehat{\omega}$ be a Kähler form of $\widehat{X \times X}$. Observe that $\Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}])$ is a non-zero positive closed current of $X \times X$ supported on Δ and has the same dimension as Δ . Therefore, it equals a constant times $[\Delta]$, see, e.g., [4]. By normalizing $\widehat{\omega}$, we can suppose that

$$(1.2) \quad \Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]) = [\Delta].$$

Put $\widehat{T \otimes S} = \Pi^*(T \otimes S)$ and $\Pi_j = \pi_j \circ \Pi$ ($j = 1, 2$). Then (1.1) can be rewritten as

$$(1.3) \quad T \wedge S = (\Pi_j)_*(\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]).$$

In general, when T and S are only positive closed currents, one still can define $\widehat{T \otimes S}$ as a positive closed current outside $\widehat{\Delta}$ and extend it by 0 through $\widehat{\Delta}$. We can show that $\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]$ is well-defined provided that T has a continuous super-potential. In this case, we can use (1.3) as an alternative definition of $T \wedge S$ which gives a positive closed current, see Corollary 3.5. Proposition 3.7 below shows that this definition is equivalent to the one of Dinh and Sibony.

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2 Super-potential of positive closed currents

We will recall now some basic facts and refer to [9] for details. Let X be a compact Kähler manifold of dimension n and ω be a Kähler form on X . It is well-known that the de Rham cohomology of currents and smooth forms are canonically equal (see [12, Chap. 3]). Denote them by $H^r(X, \mathbb{C})$ with $0 \leq r \leq n$. For any closed current T of degree r , denoted by $\{T\}$ its cohomology class in $H^r(X, \mathbb{C})$. Let $H^{p,p}(X, \mathbb{R})$ be the vector subspace of $H^{p,p}(X, \mathbb{C})$ spanned by the classes of closed real $2p$ -forms. Since a closed positive (p, p) -current is real, its class belongs to $H^{p,p}(X, \mathbb{R})$. If V is an analytic subset of X of dimension $n - p$, it defines a positive closed current $[V]$ of bidegree (p, p) by integration over V . Its class will be denoted by $\{V\}$ for simplicity.

Let \mathcal{C}_p be the convex cone of positive closed (p, p) -currents on X and \mathcal{D}_p be the real vector space generated by \mathcal{C}_p . Since the Kähler form ω is strictly positive, the set \mathcal{D}_p contains all real closed smooth (p, p) -forms. Let \mathcal{D}_p^0 be the subspace of \mathcal{D}_p of currents belonging to the class 0 in $H^{p,p}(X, \mathbb{R})$. We recall the notion of $*$ -norm on \mathcal{D}_p . Consider first a positive closed current S in \mathcal{D}_p . Define its $*$ -norm by

$$\|S\|_* = |\langle S, \omega^{n-p} \rangle|$$

which is equal to the mass of S . In general, since any $S \in \mathcal{D}_p$ can be written as the difference of two positive closed currents, define

$$\|S\|_* = \inf(\|S^+\|_* + \|S^-\|_*),$$

where the infimum is taken over all $S^+, S^- \in \mathcal{C}_p$ such that $S = S^+ - S^-$. By compactness property of positive closed currents, the above infimum is attained for some S^+ and S^- . We say that S_k converges to S in \mathcal{D}_p for the $*$ -topology if S_k converges to S weakly as currents and $\|S_k\|_*$ is bounded independently of k . The following result is due to Dinh and Sibony, see [9, Th. 2.4.4] and also [6, Th. 1.1].

Proposition 2.1. *There is a positive constant c such that for all $S \in \mathcal{D}_p$, there exist smooth forms $S_k \in \mathcal{D}_p$ with $k \in \mathbb{N}$ such that S_k converges weakly to S and $\|S_k\|_* \leq c\|S\|_*$ for all k .*

Let T be in \mathcal{D}_p and R be in \mathcal{D}_q^0 . By dd^c -lemma for currents (see [11, Th. 1.2.1]), there is a real $(q - 1, q - 1)$ -current U_R such that $dd^c U_R = R$. We call U_R a potential of R . Consider the following important example of R . Let V be a hypersurface of X and β_0 be a smooth form of the same cohomology class with $[V]$. Then $R = [V] - \beta_0$ is in \mathcal{D}_1^0 . One can construct an explicit potential U_R as follows. Consider the holomorphic line bundle of X associated with V and σ a holomorphic section whose divisor is V . Take a smooth Hermitian metric on this line bundle and denote by $|\cdot|$ the norm induced by this metric. By

Poincaré-Lelong formula, there is a smooth form β_1 such that

$$dd^c \log |\sigma| = [V] - \beta_1.$$

Since $\{\beta_0\} = \{V\} = \{\beta_1\}$, there is a smooth function f on X such that $dd^c f = \beta_0 - \beta_1$. The function $U_R := \log |\sigma| - f$ is a potential of R . Note that U_R is smooth outside V and if σ' is a holomorphic function on an open neighborhood W of a point of V such that its divisor is $V \cap W$, then

$$(2.1) \quad U_R(x) - \log |\sigma'| \text{ is smooth on } W.$$

Consider now a current $R \in \mathcal{D}_{n-p+1}^0$ and an $(n-p, n-p)$ -current U_R which is a potential of R . Let $\alpha = (\alpha_1, \dots, \alpha_h)$ with $h = \dim H^{p,p}(X, \mathbb{R})$ be a fixed family of real smooth closed (p, p) -forms such that the family of classes $\{\alpha\} = (\{\alpha_1\}, \dots, \{\alpha_h\})$ is a basis of $H^{p,p}(X, \mathbb{R})$. By adding to U_R a suitable closed smooth form, we can assume that $\langle U_R, \alpha_i \rangle = 0$ for $i = 1, \dots, h$. We say that U_R is α -normalized.

Definition 2.2. ([9, Def. 3.2.2]) *Let T be a current in \mathcal{D}_p as above. The α -normalized super-potential \mathcal{U}_T of T is the function defined on smooth forms $R \in \mathcal{D}_{n-p+1}^0$ and given by*

$$\mathcal{U}_T(R) = \langle T, U_R \rangle,$$

where U_R is an α -normalized smooth potential of R . We say that T has a continuous super-potential if \mathcal{U}_T can be extended to a function on \mathcal{D}_{n-p+1}^0 which is continuous with respect to the $*$ -topology. In this case, the extension is also denoted by \mathcal{U}_T .

By [9, Lem. 3.2.1], $\mathcal{U}_T(R)$ does not depend on the choice of an α -normalized U_R . And the continuity of \mathcal{U}_T does not depend on α . Observe that when $\{T\} = 0$, the α -normalized super-potential of T does not depend on α . Indeed, in this case, it is the restriction of any potential U_T of T to the set of smooth forms in \mathcal{D}_{n-p+1}^0 . Assume that T has a continuous super-potential. Take any current $S \in \mathcal{D}_q$. Let (a_1, \dots, a_h) be the coefficients of $\{T\}$ in the basis $\{\alpha\}$. Define $T \wedge S$ to be the real $(p+q, p+q)$ -current satisfying

$$(2.2) \quad \langle T \wedge S, \Phi \rangle := \mathcal{U}_T(dd^c \Phi \wedge S) + \sum_{1 \leq j \leq h} a_j \langle \alpha_j, \Phi \wedge S \rangle,$$

for any real smooth $(n-p-q, n-p-q)$ -form Φ .

3 Alternative definition for the intersection of currents

Let $X, \widehat{X \times X}, \omega, \widehat{\omega}, \Pi, \Pi_j, \pi_j, \Delta, \widehat{\Delta}$ be as in the previous sections. Consider two currents $T \in \mathcal{D}_p$ and $S \in \mathcal{D}_q$ as above with $p+q \leq n$. Let h, a_j and α_j with

$1 \leq j \leq h$ be as in the last section. From now on, assume that T is positive and has a continuous super-potential. Note that $\Pi_j = \pi_j \circ \Pi$ are submersions, for a proof see [9] or the proof of Lemma 3.2 below. Define $\widehat{T} = \Pi_1^*(T)$ and $\widehat{S} = \Pi_2^*(S)$. They are positive closed currents on $\widehat{X \times X}$. Put $\widehat{\alpha}_j = \Pi_1^*(\alpha_j)$ for $1 \leq j \leq h$.

Lemma 3.1. *The current \widehat{T} has a continuous super-potential.*

Proof. Suppose that the classes $\{\widehat{\alpha}_j\}$ are linearly dependent. Then there exist real numbers b_j with $1 \leq j \leq h$ which are not simultaneously equal to zero and a smooth form $\widehat{\gamma}$ such that $\sum_{j=1}^h b_j \widehat{\alpha}_j = d(\widehat{\gamma})$. Taking the wedge product with $\widehat{\omega}^n$ in the last equality and then using the push-forward by $(\Pi_1)_*$ give

$$(3.1) \quad \sum_{j=1}^h b_j \alpha_j \wedge (\Pi_1)_*(\widehat{\omega}^n) = d((\Pi_1)_*(\widehat{\gamma} \wedge \widehat{\omega}^n)).$$

Note that $(\Pi_1)_*\widehat{\omega}^n$ is actually a nonzero constant since $\widehat{\omega}^n$ is closed and positive. We deduce that the left-hand side of (3.1) is a non-trivial linear combination of α_j , $1 \leq j \leq h$. However this contradicts the fact that $\{\alpha_j\}$ are linearly independent. Hence, the classes $\{\widehat{\alpha}_j\}$ are linearly independent. Complete them to be basis $\widehat{\alpha}'$ of $H^{p,p}(\widehat{X \times X}, \mathbb{R})$. Let $\mathcal{U}_{\widehat{T}}$ be the $\widehat{\alpha}'$ -normalized super-potential of \widehat{T} .

Put $\alpha_T = \sum_{j=1}^h a_j \alpha_j$ and $\widehat{\alpha}_T = \Pi_1^* \alpha_T$. Remark that α_T and $\widehat{\alpha}_T$ are in the same cohomology classes with T and \widehat{T} respectively. Let $U_{T-\alpha_T}$ be a potential of $T - \alpha_T$. Then $U_{\widehat{T}-\widehat{\alpha}_T} := \Pi_1^* U_{T-\alpha_T}$ is a potential of $\widehat{T} - \widehat{\alpha}_T$. By definition, for any smooth form $\tilde{R} \in \mathcal{D}_{2n-p+1}^0(\widehat{X \times X})$, we have

$$\mathcal{U}_{\widehat{T}}(\tilde{R}) = \langle \widehat{T}, U_{\tilde{R}} \rangle = \langle \widehat{T} - \widehat{\alpha}_T, U_{\tilde{R}} \rangle = \langle U_{\widehat{T}-\widehat{\alpha}_T}, \tilde{R} \rangle$$

By our choice of potentials, the last quantity equals

$$\langle U_{T-\alpha_T}, (\Pi_1)_* \tilde{R} \rangle = \mathcal{U}_T((\Pi_1)_* \tilde{R}).$$

The continuity of \mathcal{U}_T now implies immediately the same property for $\mathcal{U}_{\widehat{T}}$. The proof is finished. \square

Thanks to Lemma 3.1, one can define $\widehat{T} \wedge \widehat{S}$ as in (2.2). Recall that $T \otimes S$ is a positive closed $(p+q, p+q)$ -current on $X \times X$ depending continuously on T and S . Its action on smooth forms can be described as follows. Let x be local coordinates of X . They induce naturally local coordinates (x, y) on $X \times X$. For a smooth form $\Phi(x, y)$ of $X \times X$, we have

$$(3.2) \quad \langle T \otimes S, \Phi \rangle = \langle T, S(\Phi(x, \cdot)) \rangle = \langle S, T(\Phi(\cdot, y)) \rangle.$$

Let Π' be the restriction of Π to $\widehat{X \times X} \setminus \widehat{\Delta}$. The current

$$\widehat{T \otimes S} = \Pi'^*(T \otimes S)$$

is well-defined and positive closed on $\widehat{X \times X} \setminus \widehat{\Delta}$ because Π' is biholomorphic. By Proposition 5.1 of [8], the mass of $\widehat{T \otimes S}$ is bounded. Hence, it can be extended by zero to be a positive closed current of $\widehat{X \otimes X}$ through $\widehat{\Delta}$, see [4, 13, 14]. We still denote by $\widehat{T \otimes S}$ the extended current. Take a smooth closed $(1, 1)$ -form $\widehat{\beta}$ with $\{\widehat{\beta}\} = \{\widehat{\Delta}\}$. Since $\widehat{\Delta}$ is a hypersurface, choose a potential $\hat{u} = U_{[\widehat{\Delta}] - \widehat{\beta}}$ of $[\widehat{\Delta}] - \widehat{\beta}$ as in Section 2. It is smooth outside $\widehat{\Delta}$ and its behaviour near $\widehat{\Delta}$ is described by (2.1). By adding a constant to \hat{u} if necessary, we can assume that $\hat{u} \leq -1$.

Lemma 3.2. *The current $\hat{u}\widehat{S}$ is well-defined. Moreover, if smooth forms $S_k \in \mathcal{D}_q$ converge to S in the $*$ -topology, then $\hat{u}\widehat{S}_k$ converge weakly to $\hat{u}\widehat{S}$.*

Proof. We prove the first assertion. For any smooth $(2n - q, 2n - q)$ -form $\hat{\eta}$ on $\widehat{X \times X}$, we will show that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is a smooth form on X . This allows us to define

$$(3.3) \quad \langle \hat{u}\widehat{S}, \hat{\eta} \rangle = \langle S, (\Pi_2)_*(\hat{u}\hat{\eta}) \rangle.$$

To see that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is smooth, we just need to work locally. Consider local coordinates $(W, x = (x_1, \dots, x_n))$ on a chart W of X . Without loss of generality, we can suppose W is diffeomorphic to the unit ball \mathbb{B}_1 in \mathbb{C}^n . Consider induced local coordinates (x, y) on $W \times W$. We have $\Delta \cap (W \times W) = \{x = y\}$. Define new local coordinates (x', y) on $W \times W$ by putting $x' := x - y$. Hence Δ is given by the equation $x' = 0$. The set $\Pi^{-1}(W \times W)$ is biholomorphic to the manifold M in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1}$ defined by

$$M = \{(x', y, [v]) : y \in \mathbb{B}_1, x' + y \in \mathbb{B}_1, [v] \in \mathbb{P}^{n-1} \text{ and } x' \in [v]\},$$

where $[v] = [v_1 : v_2 : \dots : v_n]$ denotes the homogeneous coordinates of \mathbb{P}^{n-1} . Let M_j ($1 \leq j \leq n$) be the open subset of M containing all points $(x', y, [v]) \in M$ with $v_j \neq 0$. They form an open covering of M . For $(x', y, [v]) \in M_1$, we have $x'_1 v_j = x'_j v_1$. Choose $v_1 = 1$, then $x'_j = x'_1 v_j$. We deduce that $(x'_1, v_2, \dots, v_n, y)$ are coordinates on M_1 and $\widehat{\Delta} \cap M_1 = \{x'_1 = 0\}$. Since $\Pi_2(x'_1, v_2, \dots, v_n, y) = y$, we see that

$$\begin{aligned} (\Pi_2)_*(\hat{u}\hat{\eta}) &= \int_{x'_1, v_2, \dots, v_n} \hat{u}(x'_1, v_2, \dots, v_n, y) \hat{\eta}(x'_1, v_2, \dots, v_n, y) \\ &= \int_{x'_1, v_2, \dots, v_n} \log |x'_1| \hat{\eta}(x'_1, v_2, \dots, v_n, y) \\ &\quad + \int_{x'_1, v_2, \dots, v_n} \hat{u}'(x'_1, v_2, \dots, v_n, y) \hat{\eta}(x'_1, v_2, \dots, v_n, y), \end{aligned}$$

where $\hat{u}'(x'_1, v_2, \dots, v_n, y)$ is a smooth function, see (2.1). This implies that the last integral defines a smooth form in y . It is also clear that the integral involving

$\log |x'_1|$ depends smoothly in y . The proof of the first assertion is finished. The second assertion is a direct consequence of the identity (3.3). The proof is finished. \square

Proposition 3.3. *We have $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$.*

Proof. Consider first the case where S is smooth. So $\widehat{T} \wedge \widehat{S}$ is the usual wedge product of a current with a smooth form. We then see that $\widehat{T} \wedge \widehat{S} = \Pi^*(T \otimes S) = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. Observe that the fibers of the submersion Π_1 are transverse to $\widehat{\Delta}$. Therefore, \widehat{T} has no mass on $\widehat{\Delta}$. Hence, $\widehat{T} \wedge \widehat{S}$ has no mass on $\widehat{\Delta}$. We deduce that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ in this case because $\widehat{T \otimes S}$ has no mass on $\widehat{\Delta}$ by definition.

In general, by Proposition 2.1, there is a sequence of smooth forms $S_k \in \mathcal{D}_q$ converging to S in the $*$ -topology. The first case and the continuity on S imply that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. It remains to show that the restriction $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S})$ of $\widehat{T} \wedge \widehat{S}$ vanishes. This is equivalent to say that

$$(3.4) \quad \int_{\widehat{\Delta}} \widehat{T} \wedge \widehat{S} \wedge \widehat{\Phi} = 0,$$

for any smooth form $\widehat{\Phi}$ of bidegree $2n - p - q$. By Proposition 2.1, we can write $S = S^+ - S^-$ where S^+ and S^- are approximable by smooth positive closed forms. Since $\widehat{T} \wedge \widehat{S} = \widehat{T} \wedge \widehat{S}^+ - \widehat{T} \wedge \widehat{S}^-$, we only need to verify that $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S}^\pm) = 0$. Therefore, without loss of generality, assume that $\widehat{T} \wedge \widehat{S}$ is positive. Consequently, it suffices to prove (3.4) for $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$.

Let χ be a convex increasing smooth function on \mathbb{R} such that $\chi(t) = 0$ if $t \leq -1/4$, $\chi(t) = t$ for $t \geq 1/4$ and $0 \leq \chi' \leq 1$. For each positive integer k , put

$$\hat{u}_k = \chi(\hat{u} + k) - k.$$

This is a smooth negative quasi-p.s.h. function since $\hat{u} \leq -1$. The functions \hat{u}_k decrease to \hat{u} and $-\hat{u}_k/k$ decrease to the characteristic function $\mathbf{1}_{\widehat{\Delta}}$ of $\widehat{\Delta}$ as $k \rightarrow \infty$. The first property implies that $\widehat{S} \wedge dd^c \hat{u}_k$ converges weakly to $\widehat{S} \wedge dd^c \hat{u}$, see Lemma 3.2. We also have

$$dd^c \hat{u}_k = [\chi'(\hat{u} + k)]^2 d\hat{u} \wedge d^c \hat{u} + \chi''(\hat{u} + k) dd^c \hat{u} \geq \chi''(\hat{u} + k) dd^c \hat{u} \geq -c\widehat{\omega},$$

for some positive constant c . This yields that $dd^c \hat{u}_k = (dd^c \hat{u}_k + c\widehat{\omega}) - c\widehat{\omega}$ which is the difference of two positive closed currents in the same cohomology class $c\{\widehat{\omega}\}$. We deduce that $dd^c \hat{u}_k$ is $*$ -bounded uniformly in k and then so is $\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}$ because we have

$$(3.5) \quad \|\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\|_* \leq c\|S\|_* \|dd^c \hat{u}_k\|_*,$$

for a positive constant c depending only on (X, ω) . It follows that

$$\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q} \rightarrow \widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q}$$

in the $*$ -topology. The equality (3.4) with $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$ is equivalent to

$$(3.6) \quad \langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \rangle \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Applying the formula (2.2) to $\widehat{T} \wedge \widehat{S}$ gives

$$\langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \rangle = -\frac{1}{k} \mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) - \frac{1}{k} \langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \rangle,$$

where $\widehat{\alpha}_T = \sum_{j=1}^h a_j \widehat{\alpha}_j$. The last quantity converges to 0 as $k \rightarrow \infty$ for the mass norm of $\hat{u}_k \widehat{S}$ is bounded independently of k by Lemma 3.2. On the other hand, the continuity of $\mathcal{U}_{\widehat{T}}$ gives

$$\mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) \rightarrow \mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q})$$

which is finite, as $k \rightarrow \infty$. Hence we get (3.6). The proof is finished. \square

Lemma 3.4. *The current $\hat{u}(\widehat{T} \wedge \widehat{S})$ is well-defined. Denote it by $\hat{u}\widehat{T} \wedge \widehat{S}$ for simplicity. For any closed real smooth form $\widehat{\Phi}$ of $\widehat{X \times X}$ of the right bidegree, we have*

$$(3.7) \quad \langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle = \mathcal{U}_{\widehat{T}}(dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})) + \sum_{j=1}^h a_j \langle \widehat{S}, \hat{u}\widehat{\alpha}_j \wedge \widehat{\Phi} \rangle.$$

In particular, $\langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle$ depends continuously on S .

Proof. Using the computation in the proof of Proposition 3.3, we have

$$\langle \widehat{T} \wedge \widehat{S}, \hat{u} \cdot \widehat{\omega}^{2n-p-q} \rangle = \lim_{k \rightarrow \infty} \mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) + \langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \rangle,$$

where \hat{u}_k is defined as in Proposition 3.3. The same arguments at the end of the above proposition show that the last limit is finite. The first assertion follows. Note that each smooth closed form Φ can be written as the difference of two positive closed forms. Hence it is enough to prove (3.7) for positive closed forms Φ . The computations in Proposition 3.3 still hold for Φ in place of $\widehat{\omega}^{2n-p-q}$. Hence (3.7) follows.

In order to prove the last assertion, it is enough to prove it for positive closed forms Φ by the same reason as above. Let $\{S_l\}_{l \in \mathbb{N}}$ be a sequence of currents in \mathcal{D}_q which converges to S in the $*$ -topology. Put $\widehat{S}_l = \Pi_2^*(S_l)$. It is clear that \widehat{S}_l converges to \widehat{S} in the $*$ -topology. Lemma 3.2 implies that $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ converges weakly to $dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})$ and

$$(3.8) \quad \lim_{k \rightarrow \infty} dd^c(\hat{u}_k \widehat{S}_l \wedge \widehat{\Phi}) = dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi}),$$

for any $l \in \mathbb{N}$. Applying (3.5) to S_k in place of S , we see that the mass of $dd^c(\hat{u}_k \hat{S}_l \wedge \hat{\Phi})$ is bounded independently of k and l . This combined with (3.8) yields that the $*$ -norm of $dd^c(\hat{u} \hat{S}_l \wedge \hat{\Phi})$ is bounded independently of l . We deduces that $dd^c(\hat{u} \hat{S}_l \wedge \hat{\Phi})$ converges to $dd^c(\hat{u} \hat{S} \wedge \hat{\Phi})$ in the $*$ -topology. The continuity of $\mathcal{U}_{\hat{T}}$ now implies that the right-hand side of (3.7) depends continuously on S . The proof is finished. \square

Corollary 3.5. *Define the intersection $\widehat{T \otimes S} \wedge [\hat{\Delta}]$ by putting*

$$(3.9) \quad \widehat{T \otimes S} \wedge [\hat{\Delta}] = dd^c(\hat{u} \widehat{T \otimes S}) + \widehat{T \otimes S} \wedge \beta.$$

Then $\widehat{T \otimes S} \wedge [\hat{\Delta}]$ is positive when S is positive.

Proof. We only need to prove the positivity. This property is classical since the current $[\hat{\Delta}]$ is of bidegree $(1, 1)$. We give here a proof for the sake of the reader. Fix a small open subset \widehat{W} of $X \times X$ biholomorphic to a ball. We can find a smooth function \hat{v} on \widehat{W} such that $dd^c \hat{v} = \hat{\beta}$. Hence the function $\hat{u}' = \hat{u} + \hat{v}$ satisfies $dd^c \hat{u}' = [\hat{\Delta}] \geq 0$. So \hat{u}' is p.s.h. on \widehat{W} . We then have $\widehat{T \otimes S} \wedge [\hat{\Delta}] = dd^c(\hat{u}' \widehat{T \otimes S})$ on \widehat{W} . If \hat{u}'_k is a sequence of smooth p.s.h. functions on \widehat{W} decreasing to \hat{u}' , then the last current is the limit of $dd^c(\hat{u}'_k \widehat{T \otimes S})$ which is clearly positive since it equals $dd^c \hat{u}'_k \wedge \widehat{T \otimes S}$. The proof is finished. \square

Lemma 3.6. *Let Y be a closed subset of X . Let R be a positive (p, p) -current of X and let R_k be a sequence of positive (p, p) -currents of X converging weakly to R as currents in $X \setminus Y$. Assume that R has no mass on Y and the masses of R_k converge to the one of R . Then R_k converges weakly to R in X .*

Proof. For each $\epsilon > 0$, let Y_ϵ be the set of points in X of distance less than ϵ to Y . Let χ_ϵ be a continuous function on X such that $0 \leq \chi_\epsilon \leq 1$ and $\chi_\epsilon = 1$ on $X \setminus Y_{2\epsilon}$ and $\chi_\epsilon = 0$ on \overline{Y}_ϵ . Take any continuous real form Φ on X of bidegree $n - p$. We need to prove that

$$(3.10) \quad R_k(\Phi) \rightarrow R(\Phi) \quad \text{as} \quad k \rightarrow \infty.$$

Since a continuous form can be written as the difference of two continuous positive forms, we can assume that Φ is positive. The hypothesis on R_k implies that $R_k(\chi_\epsilon \Phi)$ converges to $R(\chi_\epsilon \Phi)$. Hence in order to prove (3.10), it is sufficient to show that

$$(3.11) \quad \lim_{\epsilon \rightarrow 0} \delta_\epsilon = 0,$$

where

$$\delta_\epsilon = \limsup_{k \rightarrow \infty} \int_{\overline{Y}_{2\epsilon}} R_k(\Phi) \rightarrow 0.$$

Let $\mu_k = R_k \wedge \omega^{n-p}$ and $\mu = R \wedge \omega^{n-p}$ be the trace measures of R_k and R respectively. Observe that δ_ϵ is less than a constant times

$$\limsup_{k \rightarrow \infty} \mu_k(\overline{Y}_{2\epsilon}) = \|R\| - \liminf_{k \rightarrow \infty} \mu_k(X \setminus \overline{Y}_{2\epsilon}).$$

Since the set $X \setminus \overline{Y}_{2\epsilon}$ is an open subset of $X \setminus Y$, the last limit is greater than $\mu(X \setminus \overline{Y}_{2\epsilon})$. Hence we get

$$\limsup_{k \rightarrow \infty} \int_{\overline{Y}_{2\epsilon}} R_k(\Phi) \lesssim \|R\| - \mu(X \setminus \overline{Y}_{2\epsilon}) = \mu(\overline{Y}_{2\epsilon}).$$

The last quantity converges to zero as $\epsilon \rightarrow 0$ because μ has no mass on Y . The proof is finished. \square

Proposition 3.7. *For $j = 1$ or 2 , we have*

$$(3.12) \quad T \wedge S = (\Pi_j)_* (\widehat{T \otimes S} \wedge [\widehat{\Delta}] \wedge \widehat{\omega}^{n-1}),$$

where $T \wedge S$ is defined as in (2.2).

Proof. As explained in Introduction, the formula (3.12) holds for smooth forms T and S . We consider now the general case. We already know that $T \wedge S$ depends continuously on S for the $*$ -topology. Let $\{S_k\}_{k \in \mathbb{N}}$ be a sequence of smooth forms in \mathcal{D}_q which converges to S in the $*$ -topology. Put $\widehat{S}_k = \Pi_2^*(S_k)$ and $R_k = \widehat{uT} \wedge \widehat{S}_k$. It follows from Lemma 3.4 that the masses of R_k converge to the mass of $R = \widehat{uT} \wedge \widehat{S}$. Moreover, R_k converges to R in $\widehat{X \times X \setminus \Delta}$. Applying Lemma 3.6 to $\widehat{X \times X}$ in the place of X , R_k and R , we see that the right-hand sides of (3.12), which is defined in Corollary 3.5, also depends continuously on S for the $*$ -topology. Hence approximating S by smooth forms allows us to assume that S is smooth. Now Lemma 3.2 applied to \widehat{T} in place of \widehat{S} implies that the right-hand side of (3.12) is continuous in T . When S is smooth, it is clear that $T \wedge S$ depends continuously on T . Therefore, (3.12) holds since we can approximate T by closed smooth forms. The proof is finished. \square

References

- [1] E. BEDFORD AND B. A. TAYLOR, *A new capacity for plurisubharmonic functions*, Acta Math., 149 (1982), pp. 1–40.
- [2] A. BLANCHARD, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. (3), 73 (1956), pp. 157–202.
- [3] S. S. CHERN, H. I. LEVINE, AND L. NIRENBERG, *Intrinsic norms on a complex manifold*, in Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 119–139.

- [4] J.-P. DEMAILLY, *Complex analytic and differential geometry*, <http://www.fourier.ujf-grenoble.fr/~demailly>.
- [5] J.-P. DEMAILLY, *Courants positifs et théorie de l'intersection*, Gaz. Math., (1992), pp. 131–159.
- [6] T.-C. DINH AND N. SIBONY, *Regularization of currents and entropy*, Ann. Sci. École Norm. Sup. (4), 37 (2004), pp. 959–971.
- [7] ———, *Geometry of currents, intersection theory and dynamics of horizontal-like maps*, Ann. Inst. Fourier (Grenoble), 56 (2006), pp. 423–457.
- [8] ———, *Pull-back of currents by holomorphic maps*, Manuscripta Math., 123 (2007), pp. 357–371.
- [9] ———, *Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms*, J. Algebraic Geom., 19 (2010), pp. 473–529.
- [10] J. E. FORNÆSS AND N. SIBONY, *Oka's inequality for currents and applications*, Math. Ann., 301 (1995), pp. 399–419.
- [11] H. GILLET AND C. SOULÉ, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math., (1990), pp. 93–174 (1991).
- [12] P. GRIFFITHS AND J. HARRIS, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.
- [13] N. SIBONY, *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J., 52 (1985), pp. 157–197.
- [14] H. SKODA, *Prolongement des courants, positifs, fermés de masse finie*, Invent. Math., 66 (1982), pp. 361–376.
- [15] C. VOISIN, *Hodge theory and complex algebraic geometry. I*, vol. 76 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, english ed., 2007. Translated from the French by Leila Schneps.

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